Class 8

Last Time: Space Curves

$\vec{r} : I \to \mathbb{R}^n$

$\uparrow$ interval

Recall: limit of space curve is the component-wise limit

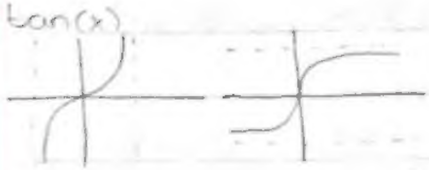
Ex. Compute $\lim\limits_{t \to \infty} \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle$

Sol. $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} \frac{1+t^2}{1-t^2} = \lim\limits_{t\to\infty} \frac{\frac{1}{t^2}+1}{\frac{1}{t^2}-1} = \frac{0+1}{0-1} = -1$

$\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty} \arctan(t) = \frac{\pi}{2}$

$\lim\limits_{t\to\infty} z(t) = \lim\limits_{t\to\infty} \frac{1-e^{-2t}}{t} = 0$

tan(x)



Hence $\lim\limits_{t\to\infty} \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle = \langle -1, \frac{\pi}{2}, 0 \rangle$

Def: A space curve $\vec{r}(t)$ is continuous at time $t=a$, if
$\lim\limits_{t\to\infty} \vec{r}(t) = \vec{r}(a)$

NB: A curve is continuous at time $t=a$ if and only if each of its components is continuous at time $t=a$.

Ex. Where is $\vec{r}(t) = \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle$ continuous?

Sol. $x(t) = \frac{1+t^2}{1-t^2}$ is cts at t iff $1-t^2 \neq 0$ i.e. $t \neq \pm 1$ i.e. $t \in (-\infty,-1) \cup (-1,1) \cup (1,\infty)$
$y(t) = \arctan(t)$ is cts for $t \in (-\infty, \infty)$

$z(t) = \frac{1-e^{-2t}}{t}$ is cts on $t \in (-\infty, 0) \cup (0, \infty)$

$\therefore \vec{r}(t)$ is cts for $t \in (-\infty, 1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$

Derivatives

Def: The derivative of space curve $\vec{r}(t)$ at time $t=a$ is

$$\vec{r}\,'(a) = \left.\frac{d\vec{r}}{dt}\right|_{t=a} = \lim_{h\to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}$$

Ex. Compute $\vec{r}\,'(t)$ for $\vec{r}(t) = \langle t, t^3, \sqrt{t} \rangle$

Sol. $\vec{r}\,'(t) = \lim\limits_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \lim\limits_{h\to 0} \frac{1}{h}\left(\langle t+h, (t+h)^3, \sqrt{t+h}\rangle - \langle t, t^3, \sqrt{t}\rangle\right)$

$$= \lim_{h\to 0} \frac{1}{h} \langle h, 3t^2h + 3th^2 + h^3, \sqrt{t+h} - \sqrt{t} \rangle$$

$$= \lim_{h\to 0} \left\langle 1, 3t^2 + 3th + h^2, \frac{\sqrt{t+h} - \sqrt{t}}{h} \right\rangle$$

$$= \left\langle \lim_{h\to 0} 1, \lim_{h\to 0} (3t^2 + 3th + h^2), \lim_{h\to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} \right\rangle$$

$$= \langle 1, 3t^2, \underline{\tfrac{1}{2} t^{-\frac{1}{2}}} \rangle$$

because: $\lim\limits_{h\to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} = \lim\limits_{h\to 0} \left(\frac{\sqrt{t+h} - \sqrt{t}}{h} \cdot \frac{\sqrt{t+h} + \sqrt{t}}{\sqrt{t+h} + \sqrt{t}} \right) = \lim\limits_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{t+h} + \sqrt{t})}$

$$= \frac{1}{\sqrt{t} + \sqrt{t}} = \frac{1}{2\sqrt{t}}$$

What's really going on $(n=2)$: $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \lim_{h\to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h} \right\rangle$$

$$= \left\langle \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} \right\rangle$$

$$= \langle x'(t), y'(t) \rangle$$

Prop (Properties of the Space-Curve Derivative):

Let $\vec{r}(t)$ and $\vec{s}(t)$ be space curves in $\mathbb{R}^n$ and let $c(t)$ be a scalar function.

① $\frac{d}{dt}\left[\vec{r}(t)+\vec{s}(t)\right] = \vec{r}\,'(t)+\vec{s}\,'(t)$ * sum rule

② $\frac{d}{dt}\left[c(t)\vec{r}(t)\right] = c'(t)\vec{r}(t)+c(t)\vec{r}\,'(t)$ * scalar rule

③ $\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{s}(t)\right] = \vec{r}\,'(t)\cdot\vec{s}(t)+\vec{r}(t)\cdot\vec{s}\,'(t)$ * product rule

$\vec{r}(t) = \langle x(t), y(t)\rangle$

$\vec{s}(t) = \langle a(t), b(t)\rangle$ $[f\cdot g]'$

$\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{s}(t)\right]$ $= f'g+g'f$

$= \frac{d}{dt}\left[x(t)a(t)+y(t)b(t)\right]$

$= \frac{d}{dt}\left[x(t)a(t)\right] + \frac{d}{dt}\left[y(t)b(t)\right]$

$= \left(x'(t)a(t)+x(t)a'(t)\right)+\left(y'(t)b(t)+y(t)b'(t)\right)$

$= \left(x'(t)a(t)+y'(t)b(t)\right)+\left(x(t)a'(t)+y(t)b'(t)\right)$

$= \langle x', y'\rangle\cdot\langle a,b\rangle + \langle x,y\rangle\cdot\langle a',b'\rangle$

$= \vec{r}\,'(t)\cdot\vec{s}(t)+\vec{r}(t)\cdot\vec{s}\,'(t)$

④ $\frac{d}{dt}\left[\vec{r}(t)\times\vec{s}(t)\right] = \vec{r}\,'(t)\times\vec{s}(t)+\vec{r}(t)\times\vec{s}\,'(t)$ * cross product rule

⑤ $\frac{d}{dt}\left[\vec{r}(c(t))\right] = \vec{r}\,'(c(t))\,c'(t)$ * chain rule

Exercise: Verify each of these for space curves in $\mathbb{R}^3$

Terminology: The tangent vector to space curve $\vec{r}(t)$ at time $t=a$ is $\vec{r}\,'(a)$.

The unit tangent vector at $t=a$ is $\frac{\vec{r}\,'(t)}{|\vec{r}\,'(a)|}$

The speed of $\vec{r}(t)$ at $t=a$ is $|\vec{r}\,'(a)|$

Exercise: Prove that if $\vec{r}(t)$ has constant speed, then $\vec{r}(t)$ is orthogonal to $\vec{r}\,'(t)$ for all t.

Integrals

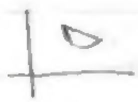
Def: The definite integral of space curve from $\lim_{n\to\infty} \sum_{k=0}^{n} f(t_k^n)\Delta t_k$

$t=a$ to b is

$$\int_{t=a}^{b} \vec{r}(t)\,dt = \int_a^b \langle x(t), y(t), z(t)\rangle\,dt$$

$$= \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

Arc Length:



The arc length of space curve $\vec{r}(t)$ from $t=a$ to b is

$s = \int_{t=a}^{b} |\vec{r}\,'(t)|\,dt$

limit of secant lines = tangent $|\vec{r}\,'(t)|$

* piece-wise linear approx of curve

$\Downarrow$

approx. length